

Meson Spectrum and Analytic Confinement

G.V. Efimov and G. Ganbold*

*Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, 141980 Dubna, Russia
(March 16, 2001)*

Abstract

A simple relativistic quantum field model with the Yukawa-type interaction is considered to demonstrate that the analytic confinement of the constituent ("quarks") and carrier ("gluons") particles explains qualitatively the basic dynamical properties of the spectrum of mesons considered as two-particle stable bound states of quarks and gluons: the quarks and gluons are confined, the glueballs represent bound states of massless gluons, the masses of mesons are larger than the sum of the constituent quark masses and the Regge trajectories of mesonic orbital excitations are almost linear.

12.40Nn; 11.10.Lm; 12.38.Aw; 14.40.-n; 14.65.-q;

Typeset using REVTeX

**Permanent address:* Institute of Physics and Technology, Mongolian Academy of Sciences, 210651 Ulaanbaatar, Mongolia

I. INTRODUCTION

The meson spectroscopy as the theory of bound states of quarks, and the phenomenology of the Regge trajectories (RTs) are important and interdependent subjects of investigation in particle physics [1–3].

At the present time, QCD is commonly considered the true theory of the strong interaction describing all processes in the hadron world, including the mesonic spectroscopy [4]. However, being a nonlinear theory with local colour gauge symmetry, QCD is quite complicated from the computational point of view and corresponding methods of calculations require great efforts in additional assumptions and ideas. In contrast to QED, simple and reliable methods of calculations are still missing in QCD. From our point of view, the main puzzle is that any acceptable and well established description of quarks and gluons explaining the hadronization on large distances, where the confinement of quarks and gluons takes place, has not been found yet. On the other hand, one may expect that a theoretical description of colorless hadrons considered as bound states of quarks and gluons, when the confinement is taken into account and the averaging over all non-observable color degrees of freedom is performed, can lead to a physical picture, where the quarks and gluons are realized in the form of some phenomenological "bricks". We suppose that a successful guess of the structure of these "bricks" in the confinement region can result, particularly, in a qualitatively correct description of the basic features of the meson spectrum. Our guess is that the analytic confinement realizes these "bricks".

Following this idea, in the present paper we consider the meson spectroscopy within the theory of bound states of quarks and gluons. Our aim is to show within a simple relativistic quantum field model that the basic features of experimentally observed meson spectrum (see, e.g. [5]) can be qualitatively explained by the analytic confinement of quarks and gluons. The basic dynamic characteristics of spectrum of all mesons considered bound states of quarks and gluons (in contrast to the relations of the SU_3 flavor symmetry) can be listed as follows:

- quarks and gluons are confined;
- glueballs are bound states of massless gluons, i.e. they are completely relativistic systems;
- the masses of mesons are larger than the sum of masses of the constituent quarks;
- the RTs of different families of mesonic orbital excitations are quite close to linear and their slopes practically coincide. It means that the slope is an universal parameter which is defined by the general nature of quark-gluon interaction.

Obviously, these characteristics cannot be obtained in the framework of any local quantum field theory, where the constituent particles, quarks and gluons, are described by standard Dirac and Klein-Gordon equations. From the common point of view the confinement plays the first role in understanding and explaining of this picture. The point is how to realize mathematically the conception of confinement in specific theoretical formalism?

The standard theoretical calculations leading to linear RTs of mesonic bound states of quarks and gluons are based on the following assumptions. First, the quarks are accepted

as ordinary fermions interacting by means of gluons. Some argumentations including lattice calculations of the nonlinear QCD gluon dynamics are given to get a particular infrared behavior of the gluon propagator ($\tilde{D}(p^2) \sim 1/p^4$ for $p^2 \rightarrow 0$) which results in a linear increasing potential between quarks in three-dimensional space $\mathbf{x} \in \mathbf{R}^3$ (see, for example, [6]). Exactly this infrared behavior is interpreted as the quark confinement.

Second, one of three-dimensional reductions of the relativistic Bethe-Salpeter equation is used. For this purpose it is necessary to overcome some mathematical problems caused by the singularity of the gluon kernel and an ambiguously defined choice of particular reduction of the relativistic two-body Bethe-Salpeter equation (see, for example, [7]). As a result a relativized Schrödinger equation is obtained in \mathbf{R}^3 -space to estimate the meson masses including higher orbital excitations. Hereby, the kinetic energy of the quarks looks like $E_{kin} = \sqrt{\mathbf{p}^2 + m^2}$ and the potential energy increases linearly. The most essential part of the two-body Hamiltonian takes the form

$$H = \sqrt{\mathbf{p}_1^2 + m_1^2} + \sqrt{\mathbf{p}_2^2 + m_2^2} + \alpha |\mathbf{r}_1 - \mathbf{r}_2|.$$

In the present paper we demonstrate another possible mechanism explaining the above mentioned characteristics of mesonic spectrum, including the origin of linear RTs. In doing so, we use a simple relativistic quantum-field model of two scalar particles (the prototypes of constituent "quarks" and intermediate "gluons") with the analytic confinement. Our approach is based on the following assumptions:

- the analytic confinement takes place, it means that propagators of "quarks" and "gluons" are entire analytic functions of the first order on the complex momentum p^2 -plane;
- the interaction is described by the Yukawa type Lagrangian;
- the coupling constant binding the "quarks" with "gluons" is small;
- final bound states of "quarks" are described by the relativistic Bethe-Salpeter equation in one-"gluon" exchange without using any three-dimensional reduction.

In the present paper we shall not discuss the origin and details of the analytic confinement. Particularly, the self-dual homogeneous vacuum gluon field in QCD results in the analytic confinement of quarks and gluons [8,9]. The model of induced quark currents based on this vacuum gluon field describes correctly the experimental data [10].

This paper is aimed to demonstrate the mathematical sketch of calculations of the two-body bound-state spectrum within the Bethe-Salpeter equation within simple QFT models, when the analytic confinement takes place in the weak-coupling regime. These simple relativistic models are based on physically transparent hypotheses and can be treated by simple analytic methods. We claim that the analytic confinement is the basic underlying "brick" leading to the qualitatively correct description of main characteristics of the meson spectra. In any case we believe that our models represent certain theoretical interest because they clarify the underlying physical principles of the meson spectrum.

For simplicity, we deal with two possible versions of the analytic confinement. In the first model the propagators of "quarks" and "gluons" are pure Gaussian exponents, i.e.

$\tilde{D}(p^2) \sim e^{-p^2/\Lambda^2}$, where the parameter $1/\Lambda$ represents the "radius" of confinement. From physical point of view this model is important because the eigenfunctions and eigenvalues of the relativistic Bethe-Salpeter equation within one-particle exchange approximation can be found explicitly and the obtained RTs are purely linear. One can say that this model can be considered a "relativistic oscillator", i.e. it is a relativistic generalization of the quantum nonrelativistic oscillator with equidistant spectra.

The second model comes from a natural physical picture of standard particles with a relevant scale of mass m . It is supposed that the analytic confinement characterized by the scale Λ makes the propagators of particles ("quarks" and "gluons") entire analytic functions. As a result there are two physical parameters: the scale of confinement Λ and the mass of quark m . An interesting problem is to investigate the dependence of the meson spectrum on the relative parameter $\mu = m/\Lambda$. We shall show that this model describes qualitatively well all above listed dynamical characteristics of meson spectra.

II. TWO-PARTICLE BOUND STATES

We consider the Yukawa model of two interacting scalar fields $\Phi(x)$ and $\varphi(x)$ described by the following Lagrangian in the Euclidean space-time as follows

$$\mathcal{L}(x) = -\Phi^+(x)S^{-1}(\square)\Phi(x) - \frac{1}{2}\varphi(x)D^{-1}(\square)\varphi(x) - g\Phi^+(x)\Phi(x)\varphi(x). \quad (1)$$

The coupling constant g is supposed to be sufficiently small.

We postulate that the analytic confinement takes place here. It means, that propagators $\tilde{S}(p^2)$ and $\tilde{D}(p^2)$ of confined particles Φ and φ are entire analytic functions in the complex p^2 -plane so that the functions $S^{-1}(z)$ and $D^{-1}(z)$ have no zeros at any finite complex z . As a result the equations for free fields

$$S^{-1}(\square)\Phi(x) = 0, \quad D^{-1}(\square)\varphi(x) = 0 \quad (2)$$

result only in the unique trivial solutions $\Phi(x) \equiv 0$ and $\varphi(x) \equiv 0$. We call this property *analytic confinement*, i.e. the corresponding particles exist only in virtual states, so Φ and φ are the *virton fields* (see [9]). One can say that these fields describe constituent particles, e.g. $\Phi(x)$ and $\varphi(x)$ represent scalar "quarks" and scalar "gluons", correspondingly.

As mentioned above, we deal with two simple versions of the analytic confinement.

First, we consider the simplest, but important case with pure Gaussian propagators

$$\begin{aligned} \tilde{S}(p^2) &= \frac{\epsilon}{\Lambda^2} e^{-\frac{p^2}{\Lambda^2}}, & S(x) &= \frac{\Lambda^2 \epsilon}{(4\pi)^2} \cdot e^{-\frac{1}{4}\Lambda^2 x^2}, \\ \tilde{D}(p^2) &= \frac{1}{\Lambda^2} \cdot e^{-\frac{p^2}{\Lambda^2}}, & D(x) &= \frac{\Lambda^2}{(4\pi)^2} \cdot e^{-\frac{1}{4}\Lambda^2 x^2}, \end{aligned} \quad (3)$$

where the parameter $\epsilon \ll 1$ implies that Φ -particle is much "heavier" than φ -particle. In some sense this model can be considered a relativistic "oscillator" because the corresponding Bethe-Salpeter equation can be solved exactly and the exact solution gives linear RTs. We call this case *the virton model*.

The second model implies that there exists a certain physical mechanism which generates analytic confinement of standard particles with the initial masses m and 0. Then, their propagators are given in more realistic forms

$$\begin{aligned}\tilde{S}(p^2) &= \frac{1}{p^2 + m^2} \cdot \left(1 - e^{-\frac{p^2 + m^2}{\Lambda^2}}\right), & S(x) &= \frac{\Lambda^2}{(4\pi)^2} \int_0^1 \frac{d\alpha}{\alpha^2} e^{-\frac{m^2}{\Lambda^2}\alpha - \frac{\Lambda^2 x^2}{4\alpha}}, \\ \tilde{D}(p^2) &= \frac{1}{p^2} \cdot \left(1 - e^{-\frac{p^2}{\Lambda^2}}\right), & D(x) &= \frac{1}{(2\pi)^2 x^2} e^{-\frac{\Lambda^2 x^2}{4}}.\end{aligned}\quad (4)$$

The confinement region is characterized by the parameter Λ . In the deconfinement limit $\Lambda \rightarrow 0$ one obtains the standard propagators of massive and massless scalar particles. Within this model we want to analyze the influence of the parameter μ on the behavior of the meson spectrum. We call this case *the scalar confinement model*.

"Two-quark" bound states can be found in the following way. Let us consider the partition function

$$Z = \iiint \delta\Phi \delta\Phi^+ \delta\phi \, e^{-(\Phi^+ S^{-1}\Phi) - \frac{1}{2}(\varphi D^{-1}\varphi) - g(\Phi^+ \Phi \varphi)}. \quad (5)$$

By integrating over φ we arrive at

$$Z = \iint \delta\Phi \delta\Phi^+ \, e^{-(\Phi^+ S^{-1}\Phi) + \frac{g^2}{2}(\Phi^+ \Phi D \Phi^+ \Phi)}. \quad (6)$$

The term $L_2[\Phi] = (\Phi^+ \Phi D \Phi^+ \Phi)$ can be transformed as

$$\begin{aligned}L_2[\Phi] &= \frac{g^2}{2} \iint dx_1 dx_2 \, \Phi^+(x_1) \Phi(x_1) D(x_1 - x_2) \Phi^+(x_2) \Phi(x_2) \\ &= \frac{g^2}{2} \int dx \iint dy_1 dy_2 \, \sqrt{D(y_1)} J(x, y_1) \delta(y_1 - y_2) \sqrt{D(y_2)} J^+(x, y_2),\end{aligned}$$

where $x_1 = x + \frac{y}{2}$, $x_2 = x - \frac{y}{2}$ and

$$\begin{aligned}J(x, y) &= \Phi^+\left(x + \frac{1}{2}y\right) \Phi\left(x - \frac{1}{2}y\right) = \Phi^+(x) e^{\frac{y}{2} \overleftrightarrow{\partial}} \Phi(x), \\ J^+(x, y) &= J(x, -y).\end{aligned}$$

Let us introduce an entire orthonormal system $\{U_Q(y)\}$, where $Q = \{n, l, \{\mu\}\}$ may be considered as a set of radial n , orbital l and magnetic $\{\mu\} = (\mu_1, \dots, \mu_l)$ quantum numbers. We have the conditions

$$\int dy \, U_Q(y) U_{Q'}(y) = \delta_{QQ'}, \quad \sum_Q U_Q(y) U_{Q'}(y') = \delta(y - y').$$

Then,

$$\begin{aligned}L_2[\Phi] &= \frac{g^2}{2} \sum_Q \int dx \, J_Q(x) \cdot J_Q(x), & J_Q(x) &= \Phi^+(x) V_Q(\overleftrightarrow{\partial}) \Phi(x), \\ J_Q^+(x) &= J_Q(x), & V_Q(\overleftrightarrow{\partial}) &= i^l \int dy \, \sqrt{D(y_1)} U_Q(y) e^{\frac{y}{2} \overleftrightarrow{\partial}}.\end{aligned}$$

By using the Gaussian functional representation we write

$$\begin{aligned} e^{L_2[\Phi]} &= e^{\frac{g^2}{2} \sum_Q \int dx J_Q(x) \cdot J_Q(x)} \\ &= \int \prod_Q \delta B_Q e^{-\frac{1}{2} \sum_Q (B_Q B_Q) + g \sum_Q \int dx B_Q(x) J_Q(x)}. \end{aligned}$$

We substitute this representation into the partition function (6) and integrate over Φ . The result reads

$$\begin{aligned} Z &= \int \prod_Q \delta B_Q e^{-\frac{1}{2} \sum_Q (B_Q B_Q) - \text{Tr} \ln(1 - g B_Q V_Q S)} \\ &= \int \prod_Q \delta B_Q e^{-\frac{1}{2} \sum_{QQ'} (B_Q [\delta_{QQ'} - \lambda \Pi_{QQ'}] B_{Q'}) + W_I[B]}. \end{aligned} \quad (7)$$

Here

$$W_I[B] = -\text{Tr} \left[\ln(1 - g B_Q V_Q S) + \frac{g^2}{2} B_Q V_Q S B_{Q'} V_{Q'} S \right]$$

is a functional, which describes the interactions of the fields B_Q .

The polarization operator $\lambda \Pi_{QQ'}$ in the one-loop approximation is

$$\begin{aligned} \lambda \Pi_{QQ'}(z) &= g^2 \text{Tr}(V_Q S V_{Q'} S) \\ &= \iint dy_1 dy_2 U_Q(y_1) \lambda \Pi(z; y_1, y_2) U_{Q'}(y_2), \\ \lambda \Pi(z; y_1, y_2) &= g^2 \sqrt{D(y_1)} S \left(z + \frac{y_1 - y_2}{2} \right) S \left(z - \frac{y_1 - y_2}{2} \right) \sqrt{D(y_2)}, \end{aligned} \quad (8)$$

where $z = x_1 - x_2$ and $\lambda = (g/4\pi\Lambda)^2$. Its Fourier transform reads

$$\begin{aligned} \lambda \tilde{\Pi}_p(y_1, y_2) &= g^2 \sqrt{D(y_1)} \int dz e^{ipz} S \left(z + \frac{y_1 - y_2}{2} \right) S \left(z - \frac{y_1 - y_2}{2} \right) \sqrt{D(y_2)}. \end{aligned} \quad (9)$$

The orthonormal system of functions $\{U_Q(y)\}$ should diagonalize the kernel (9). For this aim we consider the eigenvalue problem

$$\int dy' \lambda \tilde{\Pi}_p(y, y') U_Q(y) = E_Q(-p^2) U_Q(y), \quad (10)$$

where $E_Q(-p^2) = E_{nl}(-p^2)$, i.e. the eigenvalues are degenerated over the magnetic quantum numbers $\{\mu\}$.

If the functions $\{U_Q(y)\}$ are found, the polarization operator $\tilde{\Pi}_p(x, y)$ is diagonal on these eigenfunctions

$$\lambda \tilde{\Pi}_p(y, y') = \sum_Q E_Q(-p^2) U_Q(y) U_Q(y'). \quad (11)$$

We note that the diagonalization of $\tilde{\Pi}_p(y, y')$ is nothing else but the solution of the Bethe-Salpeter equation in the one-boson exchange approximation. To get the standard form of

the Bethe-Salpeter equation, we have to introduce new functions $U_Q(y) = \sqrt{D(y)}\Psi_Q(y)$ (see, e.g., [11]) and go to the momentum space.

So, the partition function (7) becomes

$$Z = \int \prod_Q \delta \tilde{B}_Q e^{-\frac{1}{2} \sum_Q (B_Q G_Q^{-1} B_Q) + W_I[gB]}, \quad (12)$$

$$G_Q^{-1} = G_Q^{-1}(-\square) = 1 - E_Q(\square), \quad p^2 = -\square.$$

The functional integral (12) is given over a Gaussian measure defined by the operator $G_Q^{-1}(-\square)$.

We would like to stress that this representation is completely equivalent to the initial one (5). From physical point of view, we pass on from the world containing fields Φ and ϕ to the world of bound states $\{B_Q\}$.

The fields $\{B_Q\}$ can be interpreted as the fields of particles with quantum numbers Q and masses M_Q , if the Green functions $G_Q(p^2)$ have simple poles in the Minkowski space ($p^2 = -M_Q^2$), in other words the equations

$$1 = E_Q(M_Q^2) \quad (13)$$

define the masses $M_Q = M_{nl}$ of two-particle bound states with quantum numbers $Q = (nl)$. These states are degenerated over magnetic quantum numbers $\{\mu\}$. The operator $G_Q^{-1}(-\square)$ defines the kinetic term of the field B_Q . To represent this operator in the standard form we expand it in the vicinity of $p^2 = -M_Q^2$ as follows

$$1 - E_Q(-p^2) = Z_Q(p^2 + M_Q^2) + O[(p^2 + M_Q^2)^2], \quad Z_Q = -E'_Q(-M_Q^2) > 0,$$

where the positive constant Z_Q defines the renormalization of the wave function of the field B_Q . By using the standard renormalization relation

$$\tilde{B}_Q(p) = Z_Q^{-1/2} \tilde{\mathcal{B}}_Q(p)$$

we write the kinetic term in the standard form:

$$\begin{aligned} & \left(\tilde{B}_Q^+(p) [1 - E_Q(-p^2)] \tilde{B}_Q(p) \right) \\ & = \left(\tilde{\mathcal{B}}_Q^+(p) [(p^2 + M_Q^2) + O((p^2 + M_Q^2)^2)] \tilde{\mathcal{B}}_Q(p) \right). \end{aligned}$$

The interaction functional in (12) reads

$$W_I[gB] = W_I[g_{\text{eff}}\mathcal{B}], \quad g_Q^{\text{eff}} = g Z_Q^{-1/2} = \frac{g}{\sqrt{-E'_Q(-M_Q^2)}} > 0. \quad (14)$$

It is essential, that according to (10) the effective coupling constant g_Q^{eff} does not depend on g in any explicit way.

III. THE VIRTON MODEL

Let us consider a pure Gaussian case of analytic confinement (3), where the parameter $1/\Lambda$ represents the "radius" of confinement. Due to the Gaussian character of these propagators, the polarization kernel (9) is quite simple

$$\lambda \tilde{\Pi}_p(y, y') = \lambda \frac{\Lambda^4 \epsilon^2}{2^6 \pi^2} \cdot e^{-\frac{p^2}{2\Lambda^2}} \cdot K(y, y'), \quad K(y, y') = e^{-\frac{\Lambda^2}{4}(y^2 - yy' + y'^2)} \quad (15)$$

with $\lambda = (g/4\pi\Lambda)^2$. The kernel $K(x, y)$ can be explicitly diagonalized

$$K(y, y') = \sum_Q U_Q(y) \kappa_Q U_Q(y'), \quad (16)$$

$$\kappa_Q = \kappa_{nl} = \kappa_0 \cdot \left(\frac{1}{2 + \sqrt{3}} \right)^{2n+l}, \quad \kappa_0 = \left(\frac{2\pi}{\Lambda^2(2 + \sqrt{3})} \right)^2.$$

The eigenfunctions $U_Q(y)$ are given in Appendix A.

According to (13) the mass spectrum of two-particle bound states can be found explicitly

$$M_Q^2 = M_{nl}^2 = M_0^2 + (2n + l) \cdot 2\Lambda^2 \ln(2 + \sqrt{3}), \quad (17)$$

$$M_0^2 = 2\Lambda^2 \ln \frac{\lambda_c}{\lambda}, \quad \lambda_c = \left(\frac{4(2 + \sqrt{3})^2}{\epsilon} \right)^2.$$

Thus, the Gaussian form of analytic confinement (3) leads to the purely linear and parallel RTs. The slope of RTs is defined only by the scale of the confinement region Λ and does not depend on the coupling constant λ and other dynamic constants entering the interaction Lagrangian.

Bound states exist in the weak-coupling regime $\lambda < \lambda_c$. If $\lambda \ll \lambda_c$ the size of the confinement region is remarkably larger than the Compton length of all bound states

$$r_{\text{conf}} \sim \frac{1}{\Lambda} \gg l_Q \sim \frac{1}{M_Q}.$$

In other words, the physical particles, described by fields $B_Q(x)$, and all physical transformations involving them take place within the confinement region.

IV. THE SCALAR CONFINEMENT MODEL

The propagators of this model are given by (4). In this case the eigenvalue problem (10) cannot be solved exactly, so a variational approach will be used to evaluate approximately the two-particle spectra. Further we consider only the orbital excitations, i.e. the eigenvalues for zero radial quantum number $n = 0$, so that $Q = \{0, l, \{\mu\}\}$. If $\{\Psi_Q(y)\} = \{\Psi_{l\{\mu\}}(y)\}$ is a set of normalized trial wave functions, then we have to calculate

$$\max_{\Psi_Q} \sum_{\{\mu\}} \iint dy_1 dy_2 \Psi_Q(y_1) \lambda \Pi_p(y_1, y_2) \Psi_Q(y_2) = \epsilon_l(M_l^2) \leq E_l(M_l^2) \quad (18)$$

with $p^2 = -M_l^2$. The mass of the bound state is determined by the equation (13):

$$\epsilon_l(M_l^2) = 1. \quad (19)$$

Note, the solution of this equation gives the upper bound to the mass M_l^2 , because $\epsilon_l(M^2) \leq E_l(M^2)$ for positive $M^2 > 0$.

In the present paper we choose the normalized trial wave function in the following form

$$\begin{aligned} \Psi_{l\{\mu\}}(x, a) &= C_l T_{l\{\mu\}}(x) \sqrt{D(x)} e^{-\frac{\Lambda^2}{4} ax^2}, \\ C_l &= \Lambda^{l+1} \sqrt{\frac{(1+2a)^{l+1}}{2^l(l+1)!}}, \quad \sum_{\{\mu\}} \int dx |\Psi_{l\{\mu\}}(x)|^2 = 1, \end{aligned} \quad (20)$$

where a is a variational parameter. The four-dimensional spherical orthogonal harmonics $T_{l\{\mu\}}(x)$ are defined in Appendix A. We guess that the test function in (20) should be a sufficiently good approximation to the exact one because the kernel (9) is proportional to $\sqrt{D(y)}$ and $S(y)$ is of the Gaussian type.

Let us define

$$\tilde{\Phi}_{l\{\mu\}}(k, a) = i^l \int dx e^{-ikx} \sqrt{D(x)} \Psi_{l\{\mu\}}(x, a) = \frac{C_l}{(2\pi)^l} T_{l\{\mu\}}(k) I(k^2),$$

where

$$I(k^2) = \int d^{4+2l} Y e^{-iKY} D(Y) e^{-aY^2}, \quad K, Y \in \mathbf{R}^{4+2l}, \quad k^2 = K^2.$$

Here the rotational symmetry $D = D(y^2) = D(Y^2)$ has been used. One obtains

$$\begin{aligned} \sum_{\mu} \tilde{\Phi}_{l\{\mu\}}(k, a) \tilde{\Phi}_{l\{\mu\}}(k, a) &= \frac{C_l^2 k^{2l}(l+1)}{2^{4+3l}} \left[\int_0^{u_0} du u^l e^{-uk^2/4} \right]^2, \\ u_0 &= \frac{4}{\Lambda^2(1+a)}. \end{aligned} \quad (21)$$

Substituting representations (4), (20) and (21) into (18) and after some calculations we arrive at

$$\begin{aligned} \epsilon_l(M_l^2) &= g^2 \max_a \int \frac{dk}{(2\pi)^4} \sum_{\{\mu\}} \tilde{\Phi}_{l\{\mu\}}(k, a) \tilde{G}\left(k + \frac{p}{2}\right) \tilde{G}\left(k - \frac{p}{2}\right) \tilde{\Phi}_{l\{\mu\}}(k, a) \\ &= \frac{2\lambda}{l!} \cdot \max_c \left\{ [4c(1-c)]^{l+1} \iint_0^1 dt ds e^{-(t+s)(\mu^2 - \kappa^2)} R_l(t, s, \chi) \right\} = 1. \end{aligned} \quad (22)$$

Here

$$\begin{aligned} R_l(t, s, \chi) &= \iint_0^1 du dv e^{-\kappa^2 \chi^2} (uw)^l F_l(b, \chi), \\ F_l(b, \chi) &= \int d^4 k k^{2l} e^{-k^2 b - kp(t-s)} = e^{\chi^2/b} \left(-\frac{\partial}{\partial b} \right)^l \left[\frac{1}{b^2} e^{-\chi^2/b} \right] \end{aligned}$$

FIGURES

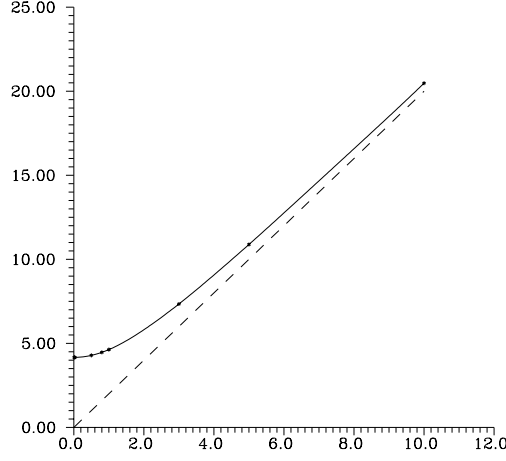


Fig. 1

The spectrum of two-particle bound state (M_0/Λ) at zero orbital angular quantum number $l = 0$ and $\lambda = 0.01$ as function of the mass $\mu = m/\Lambda$ of the "constituent" particle. The dashed curve depicts the asymptotical line 2μ .

with $p = (iM, 0, 0, 0)$ and $p^2 = -M^2$, $b = t + s + 2c(u + w)$ and

$$\lambda = \left(\frac{g}{4\pi\Lambda} \right)^2, \quad \mu = \frac{m}{\Lambda}, \quad \kappa = \frac{M_l}{2\Lambda}, \quad \chi^2 = \kappa^2(t - s)^2.$$

The mass M_l of the two-particle bound state in the one-parameter variational approximation should be determined from the variational equation (22). We have solved this equation numerically for different values of parameters λ , l and μ . The obtained results are plotted in Figs. 1-3.

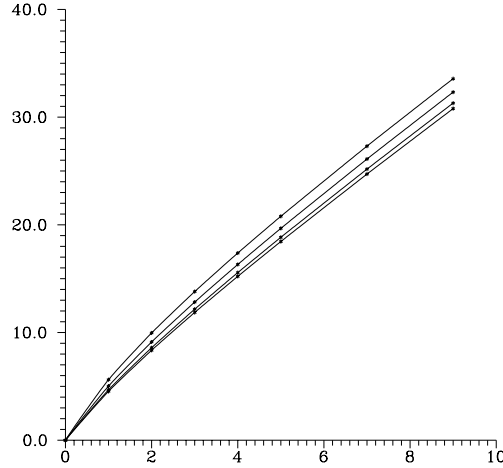


Fig. 2

The spectrum of two-particle bound state as function of the orbital angular quantum number l for several values of $\mu = m/\Lambda = \{0.05; 0.2; 1.0; 5.0\}$ at fixed effective coupling constant $\lambda = 0.01$. For convenience the difference $(M_l/\Lambda)^2 - (M_0/\Lambda)^2$ is plotted.

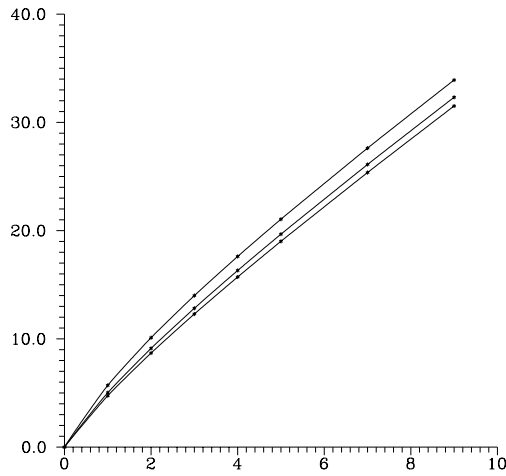


Fig. 3

The spectrum of two-particle bound state as function of the orbital angular quantum number l for several values of $\lambda = \{0.001; 0.01; 0.1\}$ at fixed $\mu = 1.0$. For convenience the difference $(M_l/\Lambda)^2 - (M_0/\Lambda)^2$ is plotted.

V. DISCUSSION

Now let us make sure that the Scalar Confinement Model correctly describes the basic features of the meson spectrum, when mesons are considered bound states of quarks and gluons. In Fig.1 we show the mass M_0/Λ of the lowest bound state B_0 with quantum numbers $Q = (0, 0) = 0$ as the function of the mass of constituent quarks $\mu = m/\Lambda$ for small coupling constant $\lambda = 0.01$.

First of all, one can conclude that there exists a bound state in the case $\mu = 0$, i.e. with $m = 0$. In other words, provided that the analytic confinement takes place, two massless particles can be coupled into a stable bound state (see also [12]). This situation cannot be realized under any circumstances in the local quantum field theory.

Second, the mass $M_0 > 2m$ for $\mu \leq 50 \div 80$, i.e. it exceeds the sum of masses of two constituent quarks. Besides, these bound states are stable. It means that these heavy mesons are relativistic systems and therefore, the nonrelativistic Schrödinger equation is not adequate tool to describe these heavy mesons. Probably the masses of real heavy quarks (see Table 1) are not sufficiently "heavy" to use nonrelativistic description. The conventional regime $M_0 < 2m$ appears for very large $\mu \geq 50 \div 80$. In the deconfinement limit $\Lambda \rightarrow 0$ our rough variational estimation (22) gives the qualitatively correct behavior (for details see Appendix B)

$$M_0 = 2m - \left(\frac{g}{4\pi m}\right)^4 16\pi m C^2,$$

i.e. we get the standard nonrelativistic (the coupling constant $g/4\pi m$ is small!) bound state under the Coulomb potential.

Quarks	Mass (<i>Mev</i>)	Meson	Mass (<i>Mev</i>)
<i>u</i>	1÷5	$\rho = u\bar{u}$	770
<i>d</i>	3÷9		
<i>s</i>	75÷170	$K^* = s\bar{s}$	900
<i>c</i>	1150÷1350	$\eta_c = c\bar{c}$	2980
<i>b</i>	4000÷4400	$\gamma = b\bar{b}$	9460

Table 1.

The experimentally observed masses of light and heavy quarks q and mesons $q\bar{q}$ [13].

In Table 1 we list the masses of quarks and mesons as bound states of corresponding quarks. One can see that the meson masses are larger than two masses of the constituent quarks. The Scalar Confinement Model describes this property of meson spectrum.

In Figures 2 and 3 we plot the Regge trajectories, i.e the squared masses of orbital excitations at different values of the small coupling constant λ and masses of constituent quarks μ . One can see that the dependence of RTs on the parameters λ and μ is not strong. Besides, the RTs are not linear, although the linearity begins in fact for $l \geq 4 \div 6$. The asymptotical behavior of the Regge spectrum for sufficiently large l can be obtained analytically and coincides with the exact solution of the Virton Model (17):

$$M_l^2 \sim l \cdot \Lambda^2 2 \ln(2 + \sqrt{3}) \quad \text{for} \quad l \rightarrow \infty. \quad (23)$$

On the other hand, the dependence M_l^2 on l is quite smooth, so that three or four lowest orbital excitations can be approximately considered almost linear. Therefore, they can be well approximated by linear trajectories on a short interval $l = 0 \div 3$.

A recent analysis of the experimental data shows (see [5]) that the RTs of different meson and baryon families are approximately linear and their slopes deviate slightly around a constant value, although the quark configurations and quantum numbers of these hadronic families are considerably different. Note, the analysed data in [5] have been obtained for low orbital momenta $l = 0 \div 3$ only.

Nevertheless, one can conclude that the slope of RTs weakly depends on specific details of hadron internal dynamics and may be considered an universal characteristic which is dictated only by the general properties of quark-gluon interactions.

Precisely this qualitative picture takes place for our models with analytic confinement. Thus, we have sufficient ground to claim that the analytic confinement realizes these general properties and leads to the approximate linearity of RTs for meson families.

Therefore, the Scalar Confinement Model should qualitatively describe the correct behavior of orbital excitations. We can compare the experimental data with our calculations. In [5] the RTs of π^- , K^- , ρ^- , ω^- , ϕ^- and K^* -meson families have been fitted on the interval $0 \leq l \leq 3$ by curves $M_l^2 = m_0 + m_1 l + m_2 l^2$ with appropriate constants m_j . We have also performed fittings of our data obtained for $l = 1, 2, 3$. By comparing corresponding

coefficients of fittings for π - and K -meson trajectories with our fit coefficients obtained at different values of parameters μ and λ we have found the possible values of the confinement scale Λ . The result is given in Table 2. One can see that Λ depends weakly on μ and λ in wide ranges. Our semiquantitative result indicates that the confinement scale parameter is around $\Lambda_{conf} \approx 500 \text{ MeV}$.

μ	λ	$\Lambda_\pi \text{ (MeV)}$	$\Lambda_K \text{ (MeV)}$
0.2	0.01	562	504
1.0	0.001	558	502
1.0	0.01	539	487
1.0	0.1	502	452
5.0	0.01	506	456

Table 2.

The scale of confinement Λ calculated from the comparison with real spectra of π - and K -meson families given in [5].

ACKNOWLEDGMENTS

The authors would like to thank I.Ya. Aref'eva, V.Ya. Fainberg and A.A. Slavnov for useful discussions.

Appendix A

Consider the kernel

$$K = K(x, y) = e^{-ax^2 + 2bxy - ay^2}, \quad a > b \quad (24)$$

with

$$\text{Tr } K = \int dy \, K(y, y) = \int dy \, e^{-2(a-b)y^2} = \frac{\pi^2}{4(a-b)^2} < \infty.$$

The eigenvalues with quantum numbers $Q = \{nl\{\mu\}\} = \{nl\{\mu_1 \dots \mu_l\}\}$ and eigenfunctions of the problem

$$\int dy \, K(x, y) U_Q(y) = \kappa_Q U_Q(x)$$

can be solved explicitly. The eigenvalues are

$$\kappa_Q = \kappa_{nl} = \kappa_0 \cdot \left(\frac{b}{a + \sqrt{a^2 - b^2}} \right)^{2n+l}, \quad \kappa_0 = \frac{\pi^2}{(a + \sqrt{a^2 - b^2})^2}. \quad (25)$$

The eigenfunctions are

$$U_Q = U_{nl\{\mu\}}(y) = N_{nl} T_{l\{\mu\}}(y) L_n^{(l+1)}(2\beta y^2) e^{-\beta y^2}. \quad (26)$$

Here $L_n^{(l+1)}(x)$ are the Laguerre polynomials and

$$\beta = \sqrt{a^2 - b^2}, \quad N_{nl} = \frac{\sqrt{2^l(l+1)}}{\pi} \cdot (2\beta)^{1+\frac{l}{2}} \cdot \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+l+2)}}$$

The functions

$$T_{l\{\mu\}}(y) = T_{l\{\mu\}}(n_y)|y|^l, \quad n_y = \frac{y}{|y|}, \quad |y| = \sqrt{y^2}$$

satisfy the conditions

$$\begin{aligned} T_{l\{\mu_1\mu_2\ldots\mu_l\}}(n) &= T_{l\{\mu_2\mu_1\ldots\mu_l\}}(n), & T_{l\{\mu\mu\mu_3\ldots\mu_l\}}(n) &= 0, \\ \sum_{\{\mu\}} T_{l\{\mu\}}(n_1) T_{l\{\mu\}}(n_2) &= \frac{1}{2^l} C_l^1((n_1 n_2)), & C_l^1(1) &= l+1, \end{aligned}$$

where $C_l^1(t)$ are the Gegenbauer polynomials and

$$\int dn \, T_{l\{\mu\}}(n) T_{l'\{\mu'\}}(n) = \delta_{ll'} \delta_{\{\mu\}\{\mu'\}} \cdot \frac{2\pi^2}{2^l(l+1)}.$$

Besides, the following relation takes place

$$\begin{aligned} \int d^4y \, T_{l\{\mu\}}(y) F(y^2) e^{-iky} &= (-i)^l T_{l\{\mu\}}(k) J(k^2), \\ J(k^2) &= \int dY e^{-iKY} F(Y^2), \quad K, Y \in \mathbf{R}^{4+2l}, \quad k^2 = K^2. \end{aligned} \quad (27)$$

Appendix B

Consider the variational problem (22) in the deconfinement limit $\Lambda \rightarrow 0$ for $l = 0$

$$\begin{aligned} 8 \left(\frac{m}{\Lambda} \right)^2 \lambda_0 \max_{0 < c < 1} & \left\{ c(1-c) \iint_0^1 dt ds \, e^{-\left(\frac{m^2}{\Lambda^2} - \frac{M_0^2}{4\Lambda^2} \right)(t+s)} \right. \\ & \cdot \left. \iint_0^1 du dv \, \frac{\exp \left\{ -\frac{M_0^2}{4\Lambda^2} \frac{(t-s)^2}{t+s+2c(u+v)} \right\}}{[t+s+2c(u+v)]^2} \right\} = 1. \end{aligned} \quad (28)$$

Here M_0 is the mass of the lowest bound state and the effective coupling constant is given as

$$\lambda_0 = \left(\frac{g}{4\pi m} \right)^2 \ll 1.$$

Going to the new variables

$$t = \frac{\Lambda^2}{2m^2}(x+y), \quad s = \frac{\Lambda^2}{2m^2}(x-y), \quad c = \frac{\Lambda^2}{m^2}\xi$$

we rewrite (28)

$$4\lambda_0 \max_{\xi} \left\{ \xi \left(1 - \frac{\Lambda^2}{m^2}\xi \right) \int_0^{\frac{m^2}{\Lambda^2}} dx e^{-\left(1 - \frac{M_0^2}{4m^2}\right)x} \cdot \iint_0^1 \frac{dudv}{[x + 2\xi(u+v)]^2} \int_{-x}^x dy e^{-\frac{M_0^2}{4m^2} \frac{y^2}{x+2\xi(u+v)}} \right\} = 1.$$

Obviously, the limit $\Lambda \rightarrow 0$ exists if $M_0 < 2m$. Besides, since $\lambda_0 \ll 1$ and $1 - \frac{M_0}{2m} \ll 1$, the main contribution into the integral over dx comes from large x , so that the inner integral over dy can be explicitly taken on the extended interval $\{-\infty, \infty\}$ without any loss of accuracy. Thus, we get

$$\frac{8m\lambda_0}{M_0} \sqrt{\frac{\pi}{1 - M_0^2/4m^2}} \cdot C = 1, \quad (29)$$

$$C = \max_{0 < \xi < \infty} \left\{ \xi \int_0^\infty dx e^{-x} \iint_0^1 \frac{dudv}{[x + 2\xi(u+v)]^{3/2}} \right\} = 0.31923....$$

By solving (29) we obtain the mass of the lowest two-particle bound state in the deconfinement limit $\Lambda \rightarrow 0$ as follows

$$M_0 = 2m - \lambda_0^2 \cdot 16\pi m C^2 + O(\lambda_0^4). \quad (30)$$

This behavior is in accordance with the conventional nonrelativistic picture of the attractive Coulomb interaction when the binding energy of two-particle bound system is negative.

REFERENCES

- [1] T. Regge, *Nouvo Cim.* **14** (1959) 951; *Nouvo Cim.* **18** (1960) 947;
- [2] G.F. Chew and S.C. Frautschi, *Phys. Rev. Lett.* **8** (1962) 41;
- [3] P.D.B. Collins, *An Introduction to Regge Theory and High Energy Physics*, Cambridge Univ. Press, Cambridge (1977);
- [4] S. Weinberg, *The Quantum Theory of Fields*, Cambridge Univ. Press, Cambridge (1995);
- [5] A. Tang and J.W. Norbury, *Phys. Rev.* **D62** (2000) 016006;
- [6] S. Godfrey, N. Isgur, *Phys. Rev.* **D32** (1985) 189;
- [7] K.M. Maung, D.E. Kahana, J.W. Norbury, *Phys. Rev.* **D47** (1993) 1182, *Phys. Rev.* **D48** (1993) 3408;
- [8] H. Leutwyler, *Phys. Lett.* **96B** (1980) 154; *Nucl. Phys.* **B179** (1981) 129;
- [9] G.V. Efimov and M.A. Ivanov, *The Quark Confinement Model of Hadrons* IOP Publishing Ltd, London, (1993);
- [10] G.V. Efimov, A.C.Kalloniatis, S.N.Nedelko, *Phys. Rev* **D59** (1999) 014026;
- [11] G.V. Efimov, hep-ph/9907483 (1999);
- [12] F.E. Close, hep-ph/0006288; M.R.Pennington, hep-ph/9811276;
- [13] Particle Data Group, April 2000 Edition.